when we raise the number 5 to the power 2, we write $5^2 = 25$. The reverse of this squaring process is called finding **a square root**. Note that 25 has two square roots 5, and -5 because $5^2 = 25$, and $(-5)^2 = 25$.

Definition 1. Principal Square Root: *The principal square root* $\sqrt{a} = b$ *means:*

1. $b^2 = a$.
2. $b \geq 0$.

Example 1. *The principal square root of* 25 *is* $\sqrt{25} = 5$.

Note 1. For any nonnegative real number x *(i.e.* $x \geq 0$*),*

$$(\sqrt{x})^2 = x$$

Definition 2. Product and Quotient Properties of Square Roots:

$$\sqrt{a \cdot b} = \sqrt{a}\sqrt{b}, \quad a \geq 0,\, b \geq 0$$

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}, \quad a \geq 0,\, b > 0$$

Example 2. *Simplify the following:*

$$\begin{aligned}\sqrt{\frac{25y^3}{9x^2}} &= \frac{\sqrt{25y^3}}{\sqrt{9x^2}} \\ &= \frac{\sqrt{25 \cdot y^2 \cdot y}}{\sqrt{9 \cdot x^2}} \\ &= \frac{\sqrt{25}\sqrt{y^2}\sqrt{y}}{\sqrt{9}\sqrt{x^2}} \\ &= \frac{5y\sqrt{y}}{3x}\end{aligned}$$

Remark 1. ‡WARNING‡ $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$.

Definition 3. Rationalizing the Denominator: *To rationalize the denominator is to multiply by a suitable factor to eliminate the square roots in the denominator. Here are some examples:*

Denominator factor	Multiply by	New Denominator
$2+\sqrt{3}$	$2-\sqrt{3}$	$(2+\sqrt{3})(2-\sqrt{3}) = 2^2 - (\sqrt{3})^2 = 4-3 = 1$
$\sqrt{5}-\sqrt{2}$	$\sqrt{5}+\sqrt{2}$	$(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2}) = (\sqrt{5})^2-(\sqrt{2})^2 = 5-2 = 3$
$1+\sqrt{x}$	$1-\sqrt{x}$	$(1+\sqrt{x})(1-\sqrt{x}) = 1^2 - (\sqrt{x})^2 = 1-x$

Example 3. *Rationalize the denominator of:* $\frac{x}{\sqrt{x}-3}$.
Solution:

$$\begin{aligned}\frac{x}{\sqrt{x}-3} &= \frac{x}{\sqrt{x}-3}\cdot\frac{\sqrt{x}+3}{\sqrt{x}+3}\\ &= \frac{x\cdot(\sqrt{x}+3)}{(\sqrt{x}-3)\cdot(\sqrt{x}+3)}\\ &= \frac{x\sqrt{x}+3x}{(\sqrt{x})^2-3^2}\\ &= \frac{x\sqrt{x}+3x}{x-9}\end{aligned}$$

Definition 4. **Other Roots:** *If $n > 0$, we say that b is an nth root of a if $b^n = a$. We next define the* **principal nth root** *of a real number a, denoted by the symbol $\sqrt[n]{a}$.*

1. *If a is positive ($a > 0$), then $\sqrt[n]{a} = b$ provided that $b^n = a$ and $b > 0$.*
2. *If a is negative ($a < 0$) and n is odd, then $\sqrt[n]{a} = b$ provided that $b^n = a$.*
3. *If a is negative ($a < 0$) and n is even, then $\sqrt[n]{a}$ is not a real number.*

Example 4. $\sqrt[3]{-64} = \sqrt[3]{(-4)^3} = -4$. *While $\sqrt[4]{-64}$ is not a real number because $n = 4$ is even and the radicand $= -64$ is negative.*

Definition 5. **Product and Quotient Properties for nth Roots:**

$$\sqrt[n]{a\cdot b} = \sqrt[n]{a}\cdot\sqrt[n]{b}$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

Definition 6. **Rational Exponent:** *For any real number a and any integer $n > 1$,*

$$a^{\frac{1}{n}} = \sqrt[n]{a}.$$

When n is even and $a < 0$, $a^{\frac{1}{n}}$ and $\sqrt[n]{a}$ are not real numbers.

Example 5. *Evaluate:*
1. $(-27)^{\frac{1}{3}} = \sqrt[3]{-27} = \sqrt[3]{(-3)^3} = -3$
2. $(\frac{1}{16})^{\frac{1}{4}} = \sqrt[4]{\frac{1}{16}} = \sqrt[4]{(\frac{1}{2})^4} = \frac{1}{2}$